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Convexity in Linear Fractional Programming Problem

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Abstract

Linear programming is a mathematical programming technique to optimize performance under a set of resource constraints as specified by organization. Linear fractional programming is a generalization of linear programming. The objective functions in linear programs are linear functions while the objective function in a linear fractional program is a ratio of two linear functions. In his paper an attempt is made to solve the convexity in linear fractional programming problem by taking CCR model, which states that the collection of all feasible solution to CCR model constitutes a convex set whose extreme points correspond to the basic feasible solutions.

Keywords: Fractional programming, CCR model, Convexity.

Introduction

Linear programming is a mathematical modeling technique designed to optimize the usage of limited resources. Successful application of linear programming exist in the areas of military, industry, agriculture, transportation, economics, health systems and even behavioral and social sciences[4], while a linear fractional programming (LFP) problem is one whose objective function has a numerator and a denominator. Several methods to solve this problem have been proposed so far [6]. Charnes and Kooper [1] have proposed a method which depends on transferring the LFP problem to an equivalent linear program.

Linear Fractional Programming

Hungarian mathematician Bela Martos formulated and considered a so called hyperbolic programming problem in the year 1960, which in the English language special literature is referred as linear fractional programming problems. In a typical case the common problem of LFP may be formulated as follows [3]:

Given objective function $Q(x) = \frac{P(x)}{D(x)} = \frac{\sum_{j=1}^{n} P_j x_J + P_o}{\sum_{j=1}^{n} d_j x_j + d_0}$ where D(x)>0

Which must be maximized (or minimized) subject to

$\sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, 2, 3, \dots \dots \dots \dots \dots \dots \dots \dots m_1$

$\sum_{j=1}^{n} a_{ij} x_j \ge b_i, \quad i = m_1 + 1, m_1 + 2, \dots \dots \dots \dots \dots \dots \dots m_2$

$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = m_2 + 1, m_2 + 2, \dots \dots \dots \dots \dots \dots \dots . m$

$x_j \ge 0, \quad j = 1,2, \dots \dots \dots \dots \dots \dots . . n$

A linear programming problem is said to be in general form if all constraints are ≤ (less than) inequalities and all unknown variables are non- negative, that is

$Q(x) = \frac{P(x)}{D(x)} = \frac{\sum_{j=1}^{n} P_j x_J + P_o}{\sum_{j=1}^{n} d_j x_j + d_0}$ → Maximize (minimize)

Subject to $\sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, 2, 3, \dots \dots \dots \dots \dots \dots \dots \dots \dots m$

$D(x) > 0, \quad \forall x \epsilon S$

Relationship with Linear programming

It is obvious that if all $d_j = 0, \quad j = 1,2, \dots \dots \dots . n$ and $d_0 = 1$ then linear fractional programming problem becomes a linear programming problem. This is a reason why we say that a linear fractional programming problem is a generalization of an linear programming problem.

Given objective function $P(x) = \sum_{j=1}^{n} P_j x_j + P_0$
Which must be maximized (minimized) subject to
$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, 3, \dots \dots \dots \dots \dots \dots \dots \dots \dots m_1$
$\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad i = m_1 + 1, m_1 + 2, \dots \dots \dots \dots \dots \dots \dots m_2$
$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = m_2 + 1, m_2 + 2, \dots \dots \dots \dots \dots \dots \dots . m$
$x_j \geq 0, \quad j = 1,2, \dots \dots \dots \dots \dots \dots . . n$
There are also a few special cases when the original LFP problem may be replaced with an appropriate LP problem.
Case I: If $d_j = 0, \quad j = 1,2, \dots \dots \dots . n, d_0 \neq 0$, then objective function Q(x) becomes a linear one

$$Q(x) = \sum_{j=1}^{n} \frac{P_j}{d_0} x_j + \frac{P_0}{d_0} = \frac{P(x)}{d_0}$$

In this case maximization (minimization) of the original objective function Q(x) may be substituted with maximization (minimization) of linear function $\frac{P(x)}{d_0}$ corresponding on the same feasible set S.
Case II: If $P_j = 0, \quad j = 1,2, \dots \dots \dots . n$, then objective function

$$Q(x) = \frac{P(x)}{D(x)} = \frac{P_0}{\sum_{j=1}^{n} d_j x_j + d_0}$$

The above equation may be replaced with function D(x). In this case maximization (minimization) of the original objective function Q(x) must be substituted with maximization (minimization) of a new objective function D(x) on the same feasible set S.
Case III: If vectors $P = P_1, P_2, \dots \dots . . P_n,$ and $d = d_1, d_2, \dots \dots d_n,$ are linearly dependent that is their exist such $\mu \neq 0, that\ P = \mu d$ the objective function
$Q(x) = \frac{P(x)}{D(x)} = \frac{\sum_{j=1}^{n} \mu d_j x_j + P_0}{\sum_{j=1}^{n} d_j x_j + d_0} = \dots \dots \dots \dots \dots \dots \dots \dots . = \mu + \frac{P_0 - \mu d_0}{\sum_{j=1}^{n} d_j x_j + d_0}$
$x_j \geq 0, \quad j = 1,2, \dots \dots \dots \dots \dots \dots . . n$
Where $D(x) > 0, \quad \forall x \epsilon S$
A linear fractional programming problem is said to be in general form if all constraints are ≤ (less than) inequalities and all unknown variables are non- negative, that is
$Q(x) = \frac{P(x)}{D(x)} = \frac{\sum_{j=1}^{n} P_J x_J + P_o}{\sum_{j=1}^{n} d_j x_j + d_0} \rightarrow$ Maximize (minimize)
Subject to $\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, 3, \dots \dots \dots \dots \dots \dots \dots \dots \dots m$
$D(x) > 0, \quad \forall x \epsilon S$

CCR (Charnes, Cooper & Rhodes) Model

The Data envelopment analysis originally proposed by Charnes, Cooper and Rhodes (1978) [2] is called the CCR model. This model allows input reducing and output increasing orientations and assumes constant returns to scale.
Ratio form of the CCR model:
Maximize $\frac{\sum_{j=1}^{n} V_j Y_{jc}}{\sum_{i=1}^{m} u_i x_{ic}}$
Subject to $\frac{\sum_{j=1}^{s} v_j y_{jk}}{\sum_{i=1}^{m} u_i x_{ik}} \leq 1, \quad k = 1,2, \dots \dots \dots . . n$
$u_i \geq \varepsilon, \quad i = 1, 2, 3, \dots \dots \dots \dots \dots \dots \dots \dots \dots m$
$v_j \geq \varepsilon, \quad j = 1, 2, 3, \dots \dots \dots \dots \dots \dots \dots \dots . s$
Where C= DMU whose technical efficiency is being measured
x_{jk} = Quantity of input I consumed by DMU k
y_{jk} = Quantity of output j produced by DMU k
u_i = Weight assigned to input i
v_j = Weight assigned to output j
ε = Very small positive number
The fractional linear programming can be written as a linear program with s+m variables and n+s+m+1 constraint. The problem is then formulated as

CCR linear model: maximize $\sum_{j=1}^{s} v_j\, y_{jc}$
Subject to $\sum_{i=1}^{m} u_i x_{ic} = 1$
$\sum_{j=1}^{s} v_j y_{jk} - \sum_{i=1}^{m} u_i x_{ik} \leq 0,$ k= 1, 2, 3……………….n
$-u_i \leq -\varepsilon,$ i= 1, 2, …………..m
$-v_j \leq -\varepsilon,$ j= 1, 2, …………s

Convexity of CCR model

Convex set: A set C in n- dimensional space is said to be convex if for any points $x^{(1)}$, $x^{(2)}$ in set C, the line segment joining these points is also in the set C [5].
Mathematically, this definition implies that $x^{(1)}$ and $x^{(2)}$ are two distinct points in C, then every point $x = \lambda x^{(2)} + (1-\lambda)x^{(1)}$, $0 \leq \lambda \leq 1$ must also be in the set C [5].
Feasible Solution: Feasible solution is any element of the feasible region of an optimization problem. The feasible region is the set of all possible solution of an optimization problem [5].
Basic feasible solution: It is one that occurs at the corner point of the feasible region in a graph [5].
Theorem: *The collection of all feasible solution to CCR model constitutes a convex set whose extreme points correspond to the basic feasible solutions.*
Let F be a set of all feasible solution of the system
AX=1, x≥0
If the set F of solutions has only one element, then F is convex set. Hence the theorem is true in this case.
Now assume that there are at least two distinct points $x^{(1)}$ and $x^{(2)}$ in F. then we have
A$x^{(1)}$= 1 for $x^{(1)} \geq 0$
A$x^{(2)}$ = 1 for $x^{(2)} \geq 0$
We only need to show that every convex combination of any two feasible solution is also a feasible solution, we define a point $x^{(0)}$ as the convex combination of $x^{(1)}$ and $x^{(2)}$. This implies that
$x^{(0)} = \lambda x^{(2)} + (1-\lambda)x^{(1)},$ $0 \leq \lambda \leq 1$
By definition F is convex if $x^{(0)}$ also belongs to F. To shoe this is true we must show that $x^{(0)}$ satisfies the system of constraints AX = 1, x≥0
Thus A $x^{(0)}$= A $\{\lambda x^{(2)} + (1-\lambda)x^{(1)}\}$
$= \lambda Ax^{(2)} + (1-\lambda)\, Ax^{(1)}$
$= \lambda.1 + (1-\lambda).1$
$= \lambda + 1 - \lambda$
= 1
Also since $0 \leq \lambda \leq 1$ and $x^{(1)} \geq 0, x^{(2)} \geq 0$, then$x^{(0)} \geq 0$. This means that $x^{(0)} \epsilon F$ and consequently F is convex.
Extreme point correspondence:
Suppose that X = $[X_B, 0]$ is a basic feasible solution where X_B is an mX1 vector s.t. for a non-singular sub matrix B of A.

$$\mathrm{B}X_B = 1$$

If possible let us suppose that x be a point of F. Such that there exist $x^{(1)}$, $x^{(2)} \epsilon F$, so that

$$x = \lambda x^{(2)} + (1-\lambda)x^{(1)}, \quad 0 < \lambda < 1$$

Let $x^{(1)} = [u_1, v_1]$ and $x^{(2)} = [u_2, v_2]$ where u_1, u_2 are mX1 vectors and v_1, v_2 are (n-m)X1 vectors then
$[X_B, 0] = \lambda[u_1, v_1]$ + (1- λ)$[u_2, v_2]$
$X_B = \lambda\, u_1 + (1-\lambda)u_2$
$0 = \lambda\, v_1 + (1-\lambda)\, v_2, \quad 0 < \lambda < 1$
Since $x^{(1)}, x^{(2)}$ are feasible solutions therefore$u_1, v_1, u_2, v_2 \geq 0$. Now $0 < \lambda < 1$ and $0 = \lambda\, v_1 + (1-\lambda)\, v_2$. Therefore we must have$v_1 = v_2 = 0$. Thus $x^{(1)} = [u_1, 0]$ and $x^{(2)} = [u_2, 0]$. Again since $x^{(1)}$ *and* $x^{(2)}$ satisfy AX = 1, we have $Bu_1 = 1$ and Bu_2=1. Also since BX_B = 1 and since expression of 1 as linear combination of basis vectors must be unique, therefore $u_1 = u_2 = X_B$
Hence $x^{(1)} = x^{(2)} = x$. This is contradiction for$x^{(1)} \neq x^{(2)}$. Hence u is an extreme point of F.

Conclusion

In this paper, we have discussed linear fractional programming and its relationship with linear programming. Also we have proved the convexity of linear fractional programming problem. For proving the

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convexity we have considered CCR model in primal form which states that the collection of all feasible solution to CCR model constitutes a convex set whose extreme points correspond to the basic feasible solutions.

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